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PROBLEMS IN THE SPREADING AND EXTRUSION OF A LAYER OF NON-LINEARLY VISCOUS FLUID*

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In the approximation of lubrication theory /1/, an equation is derived for the non-steady flow of a thin layer of a power-law liquid along a horizontal plane, when different conditions are assumed to hold at the upper boundary of the layer. Among the problems considered are the spread of an initially localized inhomogeneity, jet overflow, and spreading (extrusion) of a semi-infinite layer. Selfsimilar solutions are constructed for these problems describing waves propagating at finite velocities in the steady thickness region of the layer. Such situations occur in problems of tectonic physics /2/, glaciomechanics /3, 4/ and polymer technology /5, 6/. The more general problem of a layer flowing along a pliable base /2/ can be handled in exactly the same way.

1. Adopting assumptions similar to those customary in the hydrodynamic theory of lubrication /1/, we shall derive an equation to describe the slow non-steady flow of a layer of incompressible viscous heavy fluid along a horizontal plane. The z axis is directed vertically upward from the plane, so that the surface of the layer, of variable thickness h, is represented by an equation $z = h(x, y, t)$. Applied to this surface is a normal pressure $P_0(x, y, t)$, either a) through a non-expandable flexible film or b) under conditions permitting free horizontal displacements (no shearing stresses).

It will be convenient to work in non-dimensional variables:

$$X = x/L,\ Y = y/L,\ Z = z/H,\ \mathbf{V} = \mathbf{v}/U,\ \alpha = H/L \ll 1$$

(H and L are the characteristic layer thickness and the characeristic horizontal scale, respectively, and U is a characteristic horizontal velocity).

In the thin-layer approximation the velocity vector is almost parallel to the supporting plane ($V_z \sim \alpha$), and the derivatives with respect to x and y in the expressions for the rates of strain may be neglected compared with the derivatives with respect to z:

$$\begin{gathered}\varepsilon_{xz} = UH^{-1}V_x'\,[1 + O(\alpha^2)],\ \ \varepsilon_{yz} = UH^{-1}V_y'\,[1 + O(\alpha^2)] \\ \varepsilon_0 = UH^{-1}\sqrt{(V_x')^2 + (V_y')^2}\,[1 + O(\alpha)],\ V_x' = \partial V_x/\partial Z,\ V_y' = \partial V_y/\partial Z\end{gathered} \tag{1.1}$$

(the quantity ε_0 is proportional to the octahedral rate of strain /6, 7/).

The simplified equations of creeping flow are

$$\begin{gathered}\partial\sigma_{xz}/\partial z - \partial p/\partial x = 0,\ \ \partial\sigma_{yz}/\partial z - \partial p/\partial y = 0 \\ \partial p/\partial z = -\rho g,\ \ \partial v_x/\partial x + \partial v_y/\partial y + \partial v_z/\partial z = 0\end{gathered} \tag{1.2}$$

where terms small to order α relative to the others have been omitted.

Eqs.(1.2) must be supplemented by rheological conditions; in the case of isothermal creepage of rocks /8/ and ice /3, 4/, as well as the flow of certain polymers /5, 6, 9/, we may take these conditions in the form characteristic for power-law liquids of the pseudoplastic type, i.e., in view of the estimates (1.1):

$$\begin{gathered}\sigma_{xz} = B\varepsilon_0^{\nu-1}\partial v_x/\partial z,\ \ \ \sigma_{yz} = B\varepsilon_0^{\nu-1}\partial v_y/\partial z \\ \varepsilon_0 = \sqrt{(\partial v_x/\partial z)^2 + (\partial v_y/\partial z)^2},\ \ \nu < 1\end{gathered} \tag{1.3}$$

At the lower boundary of the layer (contiguous with the plane) one usually assumes the no-slip condition:

$$\mathbf{v} = 0,\ z = 0$$

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In the case of a non-expandable film on the upper surface of the layer, the boundary conditions there are

$$v_x = 0,\ v_y = 0,\ p = P_0,\ (z = h) \tag{1.4}$$

In the case of free horizontal displacements of the layer surface, one replaces (1.4) by the conditions

$$\partial v_x / \partial z = 0,\ \partial v_y / \partial z = 0,\ p = P_0 \tag{1.5}$$

The solution of the third equation of (1.2), satisfying the relevant condition on the surface of the layer, corresponds to a hydrostatic pressure distribution: $p = P_0 + \rho g (h - z)$. Substituting this expression into the first two Eqs.(1.2), we obtain a system of ordinary differential equations at each point of the plane, describing the variation of the velocities v_x, v_y as functions of z (here x and y are acting as parameters). The first integral of this system has the form ($\mathbf{u} = (v_x, v_y)$ is the two-dimensional velocity field)

$$B \frac{\partial \mathbf{u}}{\partial z} \left| \frac{\partial \mathbf{u}}{\partial z} \right|^{\nu-1} = (\nabla P_0 + \rho g \nabla h) z + \mathbf{C}, \quad \nabla = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\} \tag{1.6}$$

In most problems the pressure at the layer surface is $P_0 = \text{const}$. Eq.(1.6) then takes a form in which conditions (1.4) or (1.5) are readily satisfied:

$$\partial \mathbf{u} / \partial z = K^n \mid \nabla h \mid^{n-1} \nabla h \mid z - qh \mid^{n-1} (z - qh) \tag{1.7}$$
$$(K = \rho g / B,\ n = 1 / \nu)$$

The value of q is determined by the type of boundary condition at the surface: for no slip conditions (1.4) $q = 1/2$, while in the case of a free surface (1.5) $q = 1$. Integration of Eq.(1.7) yields the horizontal velocity distribution:

$$\mathbf{u} = -(n+1)^{-1} K^n \left[(qh)^{n+1} - |qh - z|^{n+1} \right] |\nabla h|^{n-1} \nabla h \tag{1.8}$$

Integrating the equation of continuity (the last equation of (1.2)) over the layer thickness and using the fact that $v_z |_{z=h} = \partial h / \partial t$, we obtain

$$\partial h / \partial t = -\nabla \int_0^H \mathbf{u}\, dz \tag{1.9}$$

Substituting the expression for the velocity field (1.8) into (1.9) we obtain the final evolution equation for the layer thickness:

$$\partial h / \partial t = \beta \nabla [h^{n+2} \mid \nabla h \mid^{n-1} \nabla h] \tag{1.10}$$
$$\beta = b_n K^n,\ b_n = q^{n+1} (n+2)^{-1}$$

(the factor b_n depends on the type of condition assumed at the surface).

Eq.(1.10) takes a particularly simple form if n is an odd integer. In particular, when $n = 1$ we obtain a well-known equation of the non-steady filtration type, describing the spread of a Newtonian liquid /2, 10/. When $n > 1$ Eqs.(1.10) become degenerate not only when h vanishes, as in the non-steady filtration equation /11, 12/, but also in regions where the layer thickness h has non-trivial stationary values ($\nabla h = 0$).

The treatment of the general case of non-uniform pressure P_0 is analogous. For odd n the equation of plane-parallel flow in the x direction has the form

$$\frac{\partial h}{\partial t} = \beta \frac{\partial}{\partial x} \left[h^{n+2} \left(\frac{\partial h}{\partial x} + \frac{1}{\rho g} \frac{dP_0}{dx} \right)^n \right] \tag{1.11}$$

Note that Eqs.(1.10) and (1.11) are approximations, corresponding to vanishingly small Froude numbers $\mathrm{Fr} = U/\sqrt{gH}$. In accordance with our formulation of the problem, only non-negative solutions of equations of the type (1.10) are physically meaningful.

In exactly the same way one can consider the problem of a liquid spreading over an uneven subhorizontal surface $z = d(x, y; t)$ ($| \nabla d | = O(\alpha)$). The equation for the layer thickness will be

$$\partial h / \partial t = \beta \nabla [h^{n+2} \mid \nabla (h + d)^{n-1} \nabla (h + d)] \tag{1.12}$$

In particular, the base over which the liquid is spreading may be pliable. Suppose that the reference level $z = 0$ corresponds to the equilibrium state at constant layer thickness

H_0 and pressure P_0. Then an additional sag (rise) Δ in the surface of the base will result in a change in the load $\rho g(h-H_0)$ in the ratio of $\Delta = L(h-H_0)$ (L is an integrodifferential operator). For the simplest case of a Winkler-type base, for example, when the supporting surface of the layer $z=-\Delta$ is under conditions of local isostatic compensation relative to an underlying liquid of higher density ρ_1, we have $\Delta = (h-H_0)\rho/\rho_1$ (the typical situation in geodynamics /2/). The equation for the layer thickness is then of the same form as (1.10), but with a coefficient $\beta_* = k\beta$, where $k = (1-\rho/\rho_1)^n$.

One can also make allowance for surface tension forces. In that case the surface pressure P_0 in (1.4) and (1.5) is given by

$$P_0 = p_0 - \sigma\nabla^2 h\,[1+|\nabla h|^{3/2}]^{-1}$$

The equation of the first approximation is

$$\partial h/\partial t = \beta\nabla\,[h^{n+2}\,|\nabla(h-\gamma\nabla^2 h)|^{n-1}\nabla(h-\gamma\nabla^2 h)],\ \gamma = \sigma/(\rho g). \tag{1.13}$$

2. Let us consider a few problems for Eq.(1.10) with $n=3$, the properties of whose solutions are typical for other values of $n>1$ as well. The selfsimilar solutions constructed here for these problems also describe the asymptotic behaviour at large times of solutions of problems with non-selfsimilar initial data, when the details of the initial distribution become unimportant /13/.

An interesting case is the axially symmetric problem of the spread of a finite volume of liquid Q localized at the initial time $t=t_0$ in a small neighbourhood of the origin:

$$h|_{t=t_0} = \begin{cases} h_0(r), & r\leqslant r_0 \\ 0, & r>r_0 \end{cases} \quad (dh_0/dr\leqslant 0) \tag{2.1}$$

Eq.(1.10) becomes

$$\frac{\partial h}{\partial t} = \frac{\beta}{r}\frac{\partial}{\partial r}\left[rh^5\left(\frac{\partial h}{\partial r}\right)^3\right] \tag{2.2}$$

Dimensionality arguments imply that one should seek a solution of problem (2.2), (2.1) in the form

$$h = [Q^2/(\beta t)]^{1/9}\,\Phi(\xi),\ \xi = r/(\beta Q^7 t)^{1/18} \tag{2.3}$$

which leads to an ordinary differential equation for Φ:

$$18\,[\xi\Phi^5(\Phi')^3] + (\xi^2\Phi)' = 0 \tag{2.4}$$

with conditions

$$\Phi(\xi)\geqslant 0,\quad \Phi(\infty)=0,\quad 2\pi\int_0^\infty \xi\Phi(\xi)\,d\xi = 1 \tag{2.5}$$

where it is required that the thickness Φ and stream function $\Phi^5(\Phi')^3$ be continuous.

As in the case of other degenerate parabolic equations /11, 12/, problem (2.4), (2.5) has no classical (smooth) solution. The generalized solution is given by a function $\Phi(\xi)$ which satisfies Eq.(2.4) in a region $\xi<\xi_0$, where $\Phi(\xi_0-0)=0$, and vanishes identically when $\xi>\xi_0$. The solution of Eq.(2.4) is

$$\Phi = D\,(\xi_0^{4/3} - \xi^{4/3})^{3/7},\ D = (1372/4608)^{1/7} = 0.8411 \tag{2.6}$$

where ξ_0 is determined by the integral conditions(2.5):

$$\tfrac{3}{2}\pi D\Gamma(\tfrac{3}{2})\,\Gamma(\tfrac{10}{7})[\Gamma(\tfrac{41}{14})]^{-1}\xi_0^{18/7} = 1,\ \xi_0 = 0.821$$

The case of an arbitrary rheological exponent n is treated in the same way (see below). We note that a suitable "reference" time t_0 in the solution of (2.3) with non-selfsimilar initial data is conveniently chosen by the formula $t_0 = (\beta Q^7)^{-1}(r_0/\xi_0)^{18}$. When that is done the solution (2.3), (2.6) defines the "intermediate asymptotic behaviour" /13/ for a problem with initial data (2.1), when $t\gg t_0$.

A more complicated structure is observed in the solution of the problem with initial data $h_{t=t_0}$ such that $\nabla h|_{t=t_0}\equiv 0$ in certain subregions. Let us consider the problem of the plane-parallel spread (extrusion) of an initially semi-infinite layer, which can propagate horizontally, i.e.,

$$h|_{t=t_0} = \begin{cases} H_0, & x\leqslant -l_2 \\ h_0(x), & l_1>x>l_2 \quad (dh_0/dx<0) \\ 0, & x\geqslant l_1 \end{cases} \tag{2.7}$$

and under the conditions

$$h|_{x=\infty} = 0,\ h|_{x=-\infty} = H_0 \ (\text{ or } h|_{x=0} = H_0) \tag{2.8}$$

A selfsimilar solution of this class of problems for the equation

$$\frac{\partial h}{\partial t} = \beta \frac{\partial}{\partial x}\left[h^5\left(\frac{\partial h}{\partial x}\right)^3\right] \tag{2.9}$$

has the form

$$h = H_0 f(\xi),\ \xi = (x - x_0) / (\beta H_0^7 t)^{1/4} \tag{2.10}$$

The function $f(\xi)$ is found by solving the ordinary differential equation

$$4\,[f^5 (f')^3]' + \xi f' = 0 \tag{2.11}$$

with conditions

$$f(\infty) = 0;\ f(-\infty) = 1 \ (\text{ or } f(0) = 1) \tag{2.12}$$

assuming in addition that the functions f and $f^5 (f')^3$ are continuous..

Eq.(2.11) is invariant under the transformation group

$$F(\xi, \lambda) = \lambda^{-4/7} f(\lambda\xi) \tag{2.13}$$

Changing to variables

$$\tau = \ln|\xi|,\quad \varphi(\tau) = \xi^{-4/7} f,\quad \psi = \xi^{3/7}\frac{df}{d\xi} = \frac{4}{7}\varphi + \frac{\partial\varphi}{d\tau}\,\mathrm{sign}\,\xi$$

we obtain an equation for $\psi(\varphi)$:

$$\frac{d\psi}{d\varphi} = -\frac{5}{3}\frac{\psi}{\varphi} - \frac{1}{7\psi - 4\varphi}\left(\frac{11}{3}\varphi + \frac{7}{12\varphi^5\psi}\right) \tag{2.14}$$

whose field of integral curves in the half-plane $\varphi \geqslant 0$ is shown in Fig.1.

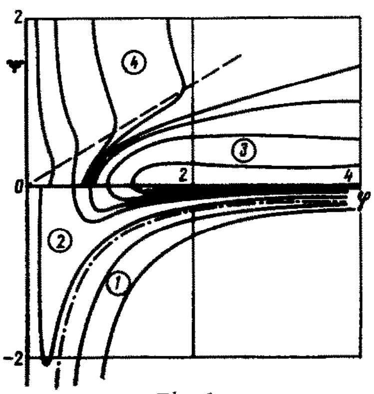

Fig.1

The integral curves of the "general position" which approach the ψ axis in the lower quadrant may be divided into two classes, depending on which terms in the equation predominate as $\varphi \to 0$. The singular integral curve separating these classes (the dash-dot curve in Fig.1) is obtained when the order of both singular terms on the right of (2.14) is the same, i.e., $\psi \approx -C_0\varphi^{-4/3}$, and the equation yields $C_0^3 = 1/4$. The corresponding solution of Eq.(2.11) has the following asymptotic representation near the leading front of the perturbation, i.e., the point ξ_1 at which $f(\xi)$ vanishes:

$$f(\xi) = \xi_1^{4/7}\,[{}^7/_3 C_0 (\xi_1 - \xi)]^{3/7} \tag{2.15}$$

Note that in this situation the stream function vanishes as required $f^5 (f')^3 \to 0$ as $\xi \to \xi_1 - 0$.

The integral curves of class *1* lie beneath this curve and correspond to the asymptotic form of Eq.(2.14) $d\psi / d\varphi = -{}^5/_3\psi\varphi^{-1} + O(1)$, so that $\psi \sim -C\varphi^{-5/3}$ as $\varphi \to 0$; consequently, the expression tends to a non-zero constant as $\xi \to \xi_1 - 0$.

The integral curves of class *2* approach the φ axis vertically, and the corresponding asymptotic form of the equation is $48\psi\,(d\psi / d\varphi) = 7\varphi^{-6}$; in the neighbourhood of the point $(\varphi_*, 0)$ the curve is described by the equation

$$\psi^2 = ({}^7/_{24})\,(\varphi - \varphi_*) / \varphi_*^6 \tag{2.16}$$

but for large φ the curves of class *2* approach the singular integral curve asymptotically from above. Thus, only the singular curve can give us a banch of an admissible solution of Eq.(2.11) for $\xi > 0$. As $\varphi \to \infty$, corresponding to $\xi \to 0$, the integral curves admit of the asymptotic representation

$$|\psi| = C_\infty\varphi^{-1/3} + o(\varphi^{-1/3}),\quad C_\infty^{(0)} \approx 0.013 \tag{2.17}$$

The behaviour of the solutions of Eq.(2.11) with negative derivative f' at $\xi < 0$ is determined by the integral curves of Eq.(2.14) situated in the upper quadrant, which leave the region of large φ values according to the asymptotic law (2.17) (curves of class *3*).

They approach the φ axis vertically, satisfying (2.16), while at the same time

$$f(\xi) - f(\xi_*) \approx m(\xi - \xi_*)^{4/3}, \quad m < 0 \tag{2.18}$$

in the neighbourhood of the point $\xi_* < 0$ at which $f' = 0$.

At this point $\xi_* = \xi_2$ for the selected monotonic solution of Eq.(2.11), the effective "diffusion coefficient" $\beta h^5 (\partial h/\partial x)^2$ of Eq.(2.9) vanishes, and therefore the front of the depression wave reaches the uniform state $h = H_0$ at a finite velocity, and instead of continuing the branch of the integral curve in Fig.1 into the lower quadrant (along a curve of class of class 2), the solution takes the constant value $f \equiv 1$ at $\xi < \xi_2$ (at the same time continuing to satisfy the flow continuity condition at the point of degeneration).

The curves of class 4, which emanate vertically from the points of the φ axis at $\varphi < \varphi_c \approx 0.882$ with asymptote (2.16), intersect the straight line $\psi = {}^4/_7\varphi$ with vertical tangent and, changing direction, go off to infinity along the ψ axis. They correspond to rapidly increasing, unbounded solutions of Eq.(2.11).

When actually constructing the selected solution (the lower solid curve on the right-hand side of Fig.2) it is convenient to use the group invariance property (2.13). Numerical methods can be used to find a solution $F_1(\xi)$ of the Cauchy problem for Eq.(2.11) in the region $\xi < 1$ with initial data for F_1 and F_1' at $\xi = 1 - \varepsilon (\varepsilon = 10^{-6} \quad 10^{-7})$ satisfying the asymptotic formula (2.15). This solution is monotonic up to the point $\xi_* \approx -3.175$, at which $F_1' = 0$. Taking $\lambda = [F_1(\xi_*)]^{7/4} = 2.9694$, we obtain the desired solution $f(\xi) = F_1(\lambda\xi) / F_1(\xi_*)$, with $\xi_1 = 1/\lambda = 0.3367$; $\xi_2 = \xi_* / \lambda = -1{,}069$. When $\xi > \xi_1$ we have $f(\xi) = 0$, but when $\xi < \xi_2$ we assume $f \equiv 1$.

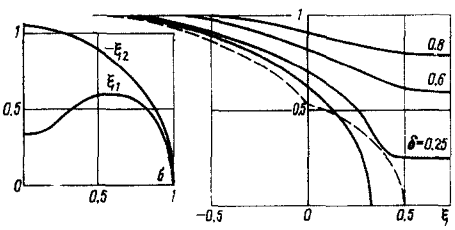

Fig.2

For the best agreement of the solution of a Cauchy problem (2.3), (2.7), satisfying initial data of a fairly general type, with the asymptotic selfsimilar solution (2.10), the free parameters t_0, x_0 should be chosen so that

$$t_0 = \beta H_0^7 \left(\frac{l_1 + l_2}{\xi_1 - \xi_2}\right)^4, \quad \int_{x_0}^{l_1} h_0(x)\,dx = \int_{-l_2}^{x_0} [H_0 - h_0(x)]\,dx \tag{2.19}$$

Numerical experiments indicate that the deviation of the non-selfsimilar solution from the selfsimilar asymptotic form for constant $\xi \in (\xi_2, \xi_1)$ decreases as least as rapidly as $(t_0/t)^2$.

In exactly the same way one can consider problems with initial data that differ from (2.7) by assuming that the layer thickness is constant over the positive half-axis (for $x > l_1$): $h = H_+ = \delta H_0 \ (0 < \delta < 1)$ (in the limit of $l_1, l_2 \to 0$ one obtains the problem of the "disintegration of an arbitrary discontinuity"). The selfsimilar intermediate asymptotic form of the solution has the same form (2.10); the leading and trailing edges of the spreading wave reach a steady at finite velocities, according to the asymptotic law $X_+ = \xi_1 (\beta H_0^7 t)^{1/4}$, $X_- = \xi_2 (\beta H_0^7 t)^{1/4}$ (unlike the analogous problem for a layer of a Newtonian liquid).

A representative phase trajectory of Eq.(2.14), joining two steady levels, will consist of two integral curves: one of class 2 for $\xi > 0$ and one of class 3 for $\xi < 0$. The corresponding solutions of Eq.(2.11) are constructed with allowance for the asymptotic representation (2.18). Graphs of the solutions for a few δ values are shown on the right-hand side of Fig. 2; the inset on the left shows curves, plotted against δ, of the numbers ξ_1 and ξ_2 determining the propagation of the leading and trailing edges of the spread wave.

In various real-life problems, particularly in geophysics, the conditions on the surface of the layer are mixed: over part of the surface the flow takes place, as it were, "under a lid", while the rest of the surface is free (and even more complex combinations may occur). As follows from Sect.1, this gives rise to a spread equation with variable (discontinuous) coefficients. Thus, under conditions of plane-parallel selfextrusion from beneath a film $x \leqslant 0$, Eq.(1.11) must be replaced in the region with free surface $x > 0$ by the equation (with $P_0 = \text{const}$)

$$\frac{\partial h}{\partial t} = \beta \frac{\partial}{\partial x}\left(h^{n+2}\left|\frac{\partial h}{\partial x}\right|^{n-1}\frac{\partial h}{\partial x}\right) \times \begin{cases} 1, & x<0 \\ 2^{n+1}, & x>0 \end{cases} \tag{2.20}$$

while at the point $x=0$ a matching condition must be satisfied (continuity of flow): $(\partial h/\partial x)_{-0} = 2^{1+1/n}\,(\partial h/\partial x)_{+0}$.

The corresponding selfsimilar solution of problem (2.9), (2.8) (in the case of conditions at $x=\pm\infty$) for $n=3$ is expressed in the form (2.10) and represented by the dashed curve in Fig.2.

More complex extrusion problems are obtained if the applied pressure in equations of type (1.11), (2.20) and their multidimensional generalizations is allowed to vary. In particular, the variation of the surface pressure P_0 may be due to the finite stiffness of the covering film layer (plate), a membrane effect, i.e, in equations of type (1.11) $P_0 = p_0 + \Lambda h$, where Λ is a differential operator.

3. We note that the basic spread Eq.(1.10) also holds for dilatant power-law liquids, when $\nu > 1$ in (1.3) (rheological exponent $n<1$). For this class of liquids Eq.(1.10) is identical with a special case of the turbulent filtration equations for a gas /14, 15/:

$$c\partial\rho / \partial t = \operatorname{div}\left[\left|\operatorname{grad}(\rho^k)\right|^{n-1}\operatorname{grad}(\rho^k)\right] \tag{3.1}$$
$$(1/2 \leqslant n \leqslant 1)$$

when the exponent of the polytropic equation of the state $\gamma = k-1$ is related to the degree of turbulence n by the equation $\gamma = 1 + 2/n$. A detailed investigation of selfsimilar solutions of Eq.(3.1) may be found in /15/, and the conclusions carry over in many cases to the spread of pseudoplastic liquids $(n>1)$. Properties associated with strong degeneration are preserved in dilatant liquids; for example, perturbations propagate at a finite velocity in the region where $h=0$. On the other hand, there is no weak degeneration (with respect to ∇h), and therefore perturbations build up to non-zero steady states at infinite velocities (unlike the basic case considered above of a pseudoplastic liquid).

We shall construct selfsimilar solutions satisfying zero initial data, for arbitrary n and a power law of variation of the total volume due to an influx concentrated on the axis $r=0$

$$2\pi \int_0^\infty h(r,t)\,r\,dr = Dt^\lambda.$$

These solutions may be treated as the non-steady overflow of a jet of liquid on a supporting plane. They have the form

$$h = \left[\frac{D^{n+1}t^{\lambda(n+1)-2}}{\beta^2}\right]^{1/(5n-3)} \Phi(\xi), \quad \xi^{5n+3} = \frac{r^{5n+3}}{\beta D^{2n+1}t^{(2n+1)\lambda+1}}$$

where Φ satifies the equation

$$(5n+3)\left[\xi\Phi^{n+2}(-\Phi')^n\right]' = \left[(2n+1)\lambda+1\right]\xi^2\Phi' - \left[\lambda(n+1)-2\right]\xi\Phi$$

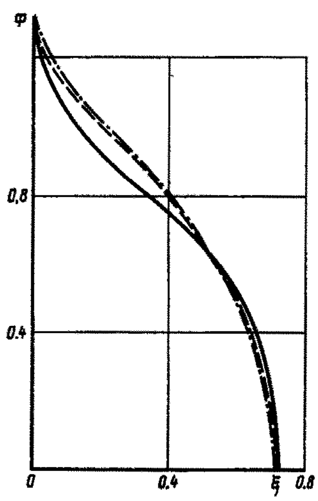

Fig.3

On the assumption that the discharge is fixed $(\lambda=1)$, the functions have the form shown in Fig.3 for $n=1$ (the solid curve), $n=3$ (the dashed curve) and $n=5$ (the dot-dash curve). In selfsimilar variables, when $n \geqslant 2.5$ one has a "quasi-universal" spread profile, distinguished only by a singularity at the front.

The derivation of Eq.(1.10) and its generalized versions carries over directly to spread and extrusion problems for more general Reiner-Rivlin liquids /6, 7, 9/. For example, suppose that the rheological equation for plane-parallel flow of a layer is

$$\gamma' = G(\tau) \quad (\gamma' = \partial v_x/\partial z, \quad \tau = \sigma_{xz})$$

Then the analogue of Eqs.(1.10) and (1.11) takes the form

$$\partial h/\partial t = -\partial Q_x/\partial x$$

$$Q_x = \int_0^h v_x\,dz = \int_0^h G\left[\left(\frac{dP_0}{dx} + \rho g\frac{\partial h}{\partial x}\right)(z-qh)\right](h-z)\,dz$$

As before, in the case of a free surface $q=1$, and for flow "under a flexible lid" $q=1/2$.

In particular, the equations for two widely used models are as follows /9/:
the Ellis-de Haven model $(G(\tau) = (A + c\,|\tau|^{n-1})\,\tau)$:

$$\frac{\partial h}{\partial t}=\frac{\partial}{\partial x}\left[\left(\varkappa h^3+\beta h^{n+2}\left|\frac{\partial h}{\partial x}\right|^{n-1}\right)\frac{\partial h}{\partial x}\right]$$

and the Prandtl-Eyring model ($G(\tau)=B\,\mathrm{sh}\,(\tau/\tau_*)$):

$$\frac{\partial h}{\partial t}=\frac{B}{R}\frac{\partial}{\partial x}\left[h\left(\frac{\partial h}{\partial x}\right)^{-1}\mathrm{ch}\left(Rh\frac{\partial h}{\partial x}\right)-\left(R\frac{\partial h}{\partial x}\right)^{-1}\mathrm{sh}\left(Rh\frac{\partial h}{\partial x}\right)\right],\quad R=\rho g/\tau_*$$

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